EE 132A Midterm Solution
Winter 2008

Inst: Dr. C.W. Walker

Problem	Points	Score
1	12	
2	12	
3	16	
4	16	
5	14	
6	14	
7	16	
Total	100	

Instructions and Information:

1) Print your name and student number at the top of the page.

2) Make sure your exam has 7 problems. Do not write any work to be graded on the back of the pages. There is extra workspace at the back of the exam.

3) This is a closed book exam. You may use one sheet of notes (front and back). You may also use a calculator.

4) Partial credit will be given but you must **show your work where appropriate or justify your answers to receive any credit.**

5) **Circle or box your final answers.**

Problem 1. Compute the power in the following deterministic signals (note a power $\to \infty$ is possible):

a. $x_1(t) = A\cos(2\pi f_0 t + \theta)$.

Solution.

$$\begin{aligned} P_{x_1} &= \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} |x_1(t)|^2 dt = \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} A^2\cos^2(2\pi f_0 t + 2\theta)dt \\ &= \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} \frac{A^2}{2}dt + \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} \frac{A^2}{2}\cos(4\pi f_0 t + \theta)dt \\ &= \frac{A^2}{2}. \end{aligned}$$

b. $x_2(t) = t^{-1/4}\ u(t)$.

Solution.

$$\begin{aligned} P_{x_2} &= \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} |t^{-1/4}u(t)|^2 dt = \lim_{T\to\infty} \frac{1}{T}\int_{0}^{T/2} t^{-1/2}dt \\ &= \lim_{T\to\infty} \sqrt{\frac{2}{T}} = 0. \end{aligned}$$

Problem 2. Compute the Fourier transform of the following (simplify your answer):

a. $x_1(t) = \prod(t - 1/2) + \prod(t + 1/2)$.

Solution. Observe that

$$x_1(t) = \prod(t - 1/2) + \prod(t + 1/2) = \prod(t/2).$$

Using $\prod(t) \longleftrightarrow \text{sinc(f)}$ we get using the scaling property

$$X_1(f) = 2\text{sinc(2f)}.$$

b. $x_2(t) = \bigwedge(2t - 4)$.

Solution. Observe that

$$x_2(t) = \bigwedge(2t - 4) = \bigwedge(2(t - 2)).$$

Using $\bigwedge(t) \longleftrightarrow \mathrm{sinc}^2(\mathrm{f})$ we get using the scaling property followed by the translation property

$$X_2(f) = \frac{1}{2} e^{-i4\pi f} \mathrm{sinc}^2(\mathrm{f}/2).$$

Problem 3. A DSB-modulated signal

$$x(t) = Am(t)\cos(2\pi f_c t)$$

is mixed (multiplied) with a local carrier

$$x_L(t) = \cos(2\pi f_c t + \theta)$$

and the output is passed through a LPF with bandwidth equal to the bandwidth of the message $m(t)$. The signal power at the output of the LPF is denoted P_{out}. The modulated signal power is denoted P_X.

a. Compute $\dfrac{P_{out}}{P_X}$.

Solution. Let $z(t) = x(t)x_L(t)$. Then

$$z(t) = \frac{A^2}{2} m(t)\cos(\theta) + \frac{A^2}{2} m(t)\cos(4\pi f_c t + \theta).$$

Let $y(t)$ be the result of passing $z(t)$ thru the LPF. Then,

$$y(t) = \frac{A^2}{2} m(t)\cos(\theta).$$

Now let P_m denote the power in $m(t)$. Then,

$$P_X = \frac{A^2}{2} P_m, \quad P_{out} = \frac{A^2}{4} P_m \cos^2(\theta)$$

so

$$\frac{P_{out}}{P_X} = \frac{\cos^2(\theta)}{2}.$$

b. Now suppose we add noise to $x(t)$ so that before the mixing operation we have

$$x(t) = Am(t)\cos(2\pi f_c t) + n(t)$$

where $n(t)$ represents bandpass filtered white Gaussian noise centered at $\pm f_c$ with spectral height $\frac{N_0}{2}$ and bandwidth B. After mixing and passing through the LPF we get an output consisting of signal and noise components. Define the SNR_{out} at the output as the power in the signal component divided by the power in the noise component. Similarly, define SNR_{in} at the input (before the mixing) as the power in the signal component divided by the power in the noise component for the signal at that point. Compute SNR_{out} in terms of SNR_{in}.

Solution. Write $n(t) = n_c(t)\cos(2\pi f_c t) - n_s(t)\sin(2\pi f_c t)$ we get

$$\begin{aligned} z(t) &= \frac{A^2}{2}m(t)\cos(\theta) + \frac{A^2}{2}m(t)\cos(4\pi f_c t + \theta) \\ &+ \frac{n_c(t)}{2}\cos(\theta) + \frac{n_c(t)}{2}\cos(4\pi f_c t + \theta) \\ &- \frac{n_s(t)}{2}\sin(\theta) - \frac{n_s(t)}{2}\sin(4\pi f_c t + \theta). \end{aligned}$$

We assume the bandpass filtered noise occupies the same bandwidth as $m(t)$ so at the output of the LPF we get

$$y(t) = \frac{A^2}{2}m(t)\cos(\theta) + \frac{n_c(t)}{2}\cos(\theta) - \frac{n_s(t)}{2}\sin(\theta).$$

We now make use of the fact that $E[n_c(t)n_s(t)] = E[n_c(t)]E[n_s(t)] = 0$ and $P_{n_c} = P_{n_s} = P_n$ we find

$$P_{out,signal} = \frac{A^2}{4}P_m\cos^2(\theta), \quad P_{out,noise} = \frac{P_n}{4}$$

and thus

$$SNR_{out} = \frac{A^2 P_m}{P_n}\cos^2(\theta).$$

Now

$$P_{in,signal} = \frac{A^2}{2}P_m, \quad P_{in,noise} = P_n$$

so

$$SNR_{in} = \frac{A^2 P_m}{2P_n}.$$

Therefore,

$$\frac{SNR_{out}}{SNR_{in}} = 2\cos^2(\theta).$$

Note: If the receiver has knowledge of θ so that, in effect, $\theta = 0$ in the above analysis, then $\frac{SNR_{out}}{SNR_{in}} = 2$. This is a well known result in communication systems. (See, for example, Communication Systems by Simon Haykin, Wiley,1983, page 325. Haykin does not provide the analysis we did here but does state our result. He does show, however, that the ratio of the output SNR to the channel SNR is unity. This is equivalent to what we did in class when comparing the demodulator output to a baseband system.)

Problem 4. The message signal $m(t) = 5\text{sinc}(200\text{t})$ frequency modulates the carrier $c(t) = 50\cos(2\pi f_c t)$. The modulation index is 8.

a. Write an expression for the modulated signal $x(t)$.

Solution. Note: $\text{sinc}(200\text{t}) \longleftrightarrow \frac{1}{200}\prod\left(\frac{\text{f}}{200}\right) \Rightarrow \text{W} = 100.$

$$x(t) = A\cos\left[2\pi f_c t + 2\pi k_f \int_{-\infty}^{t} m(\tau)d\tau\right].$$

Now $A = 50$ and

$$\beta(f) = \frac{k_f \max|m(t)|}{W} = \frac{k_f 5}{100} = 8 \Rightarrow k_f = 160$$

so

$$x(t) = 50\cos\left[2\pi f_c t + 1600\pi \int_{-\infty}^{t} \text{sinc}(200\tau)\text{d}\tau\right].$$

b. What is the maximum frequency deviation of the modulated signal?

Solution.

$$\Delta f_{max} = \beta_f W = 8(100) = 800 \text{ Hz}.$$

c. What is the power content of the modulated signal?

Solution.

$$P = \frac{A^2}{2} = \frac{2500}{2} = 1250 \text{ Watts.}$$

d. Find the bandwidth of the modulated signal?

Solution. Using Carson's rule we find

$$B_c = 2(\beta_f + 1_W = 2(8+1)100 = 1800 \text{ Hz.}$$

Problem 5. The phase modulated signal has the form

$$m(t) = 10\cos\left[2\pi f_c t + 2\sin(2\pi f_m t)\right]$$

where $f_c = 1$ MHz and $f_m = 1$ kHz.

a. Determine the modulation index.

Solution.

$$\beta_p = \Delta\phi_{max} = \max[2\sin(2\pi f_m t)] = 2.$$

b. Using Carson's rule determine the transmitted signal bandwidth.

Solution.

$$B_{PM} = 2(\beta_p + 1) f_m = 2(2+1)1000 = 6000 \text{ Hz.}$$

Problem 6. The wide-sense stationary process $X(t)$ has a correlation function denoted by $R_X(\tau)$ and a power spectral density denoted by $S_X(f)$. Suppose we form a new random process as

$$Y(t) = X(t) - X(t-T)$$

where T is some fixed constant.

a. Find the correlation function for $Y(t)$. Is $Y(t)$ wide-sense stationary?

Solution.

$$\begin{aligned}
R_Y(t_1,t_2) &= E[Y(t_1)Y(t_2)] = E\left[(X(t_1) - X(t_1-T))\,(X(t_2) - X(t_2-T))\right] \\
&= E\left[X(t_1)X(t_2)\right] - E\left[X(t_1)X(t_2-T)\right] \\
&- E\left[X(t_1-T)X(t_2)\right] + E\left[X(t_1-T)X(t_2-T)\right] \\
&= E\left[X(t_1-t_2)\right] - E\left[X(t_1-t_2+T)\right] \\
&- E\left[X(t_1-t_2-T)\right] + E\left[X(t_1-t_2)\right] \\
&= 2R_X(\tau) - R_X(\tau+T) - R_X(\tau-T).
\end{aligned}$$

So, $R_Y(t_1,t_2)$ does not depend on the actual values of t_1 and t_2 but only on their difference. So, $Y(t)$ is WSS and

$$R_Y(\tau) = R_X(\tau) - R_X(\tau+T) - R_X(\tau-T).$$

b. Find the power spectral density of $Y(t)$. (Note: depending on how you work part (b) it can be worked independent of part (a).)

Solution. Taking Fourier transforms we find

$$S_Y(f) = 2S_X(f) - e^{i2\pi fT}S_X(f) - e^{-i2\pi fT}S_X(f) = 2S_X(f)[1-\cos 2\pi fT].$$

You could also use

$$h(t) = \delta(t) - \delta(t-T), \quad S_Y(f) = S_X(f)|H(f)|^2 = S_X(f)H(f)H^*(f)$$

to get

$$S_Y(f) = S_X(f)\left[1-e^{-i2\pi fT}\right]\left[1-e^{i2\pi fT}\right] = 2S_X(f)[1-\cos 2\pi fT].$$

Problem 7. Consider a random process $X(t)$ defined by

$$X(t) = U \cos 2\pi f t + V \sin 2\pi f t$$

where f is a constant and U and V are random variables.

a. Show the condition

$$E[U] = E[V] = 0$$

is necessary for the random process to be stationary.

Solution.

$$E[X(t) = E[U] \cos 2\pi f t + E[V] \sin 2\pi f t$$

To be independent of t need

$$E[U] = E[V] = 0.$$

b. Show that $X(t)$ is wide sense stationary if and only if U and V are uncorrelated with equal variance σ^2.

Solution. Assume $X(t)$ is WSS. Then,

$$E[X(0)^2] = E\left[\left(\frac{1}{4f}\right)^2\right] = R_x(0) = \sigma_x^2.$$

Now

$$X(0) = U,\ X\left(\frac{1}{4f}\right) = V$$

so

$$E[U^2] = E[V^2] = \sigma_x^2 = \sigma^2.$$

Also,

$$R_x(t+\tau, t) = E[X(t+\tau)X(t)] = E[(U \cos 2\pi f(t+\tau) + V \sin 2\pi f(t+\tau))(U \cos \omega t + V \sin \omega t)]$$

$$= \sigma^2 \cos 2\pi f\tau + E[UV] \sin(4\pi f t + 2\pi f\tau).$$

To be independent of t need

$$E[UV] = 0 \Rightarrow E[UV] = E[U]E[V]$$

so U and V are uncorrelated.

Now assume that $E[UV] = E[U]E[V] = 0$ and $E[U^2] = E[V^2] = \sigma^2$. Then,

$$\mu_x(t) = 0$$

and

$$R_x(t+\tau, t) = \sigma^2 \cos \omega\tau = R_x(\tau)$$

so $X(t)$ is WSS.