Name: ________________________ On-campus or DEN student: _________

EE 567 Midterm Solution

October 23, 2018

Inst: Dr. C.W. Walker

Problem	Points	Score
1	14	
2	14	
3	16	
4	14	
5	14	
6	14	
7	14	
Total	100	

Instructions and Information:

1) Print your name and assigned class number at the top of the page.

2) Make sure your exam has 7 problems.

3) This is a closed book exam. You may use one 8.5 x 11 inch sheet of notes (front and back). You may use a self-contained calculator but not a computer. Cell phones or any device with internet capability is not permitted. **You have 1 hour and 20 minutes to take this exam**.

4) The points for each problem is shown above.

Problem 1. Pulse Coded Modulation (PCM) is to be used to encode a signal. The signal ranges between the values -2 and +2. There are 3 bits or 8 levels (hence 8 code numbers) available. The levels assigned have symmetry like we demonstrated in class. The first three sample values obtained (before quantization) are 1.9, 0.2, and -1.1, respectively.

a. Find the quantized values for the three sample values.

   Solution: We divide the range -2 to +2 into 8 segments each of length 0.5. Then,

$$1.9 \to 1.75,\ 0.2 \to 0.25,\ -1.1 \to -1.25.$$

b. Find the corresponding PCM sequences for the quantized values.

   Solution: 111 100 001

Problem 2. A DSB-modulated signal

$$x(t) = Am(t)\cos(2\pi f_c t)$$

is mixed (multiplied) with a local carrier

$$x_L(t) = \cos(2\pi f_c t + \theta)$$

and the output is passed through a LPF with bandwidth equal to the bandwidth of the message $m(t)$. The signal power at the output of the LPF is denoted P_{out}. The modulated signal power is denoted P_X.

a. Compute $\frac{P_{out}}{P_X}$.

   Solution. Let $z(t) = x(t)x_L(t)$. Then

$$z(t) = \frac{A^2}{2}m(t)\cos(\theta) + \frac{A^2}{2}m(t)\cos(4\pi f_c t + \theta).$$

   Let $y(t)$ be the result of passing $z(t)$ thru the LPF. Then,

$$y(t) = \frac{A^2}{2}m(t)\cos(\theta).$$

Now let P_m denote the power in $m(t)$. Then,

$$P_X = \frac{A^2}{2} P_m, \quad P_{out} = \frac{A^2}{4} P_m \cos^2(\theta)$$

so

$$\frac{P_{out}}{P_X} = \frac{\cos^2(\theta)}{2}.$$

b. Now suppose we add noise to $x(t)$ so that before the mixing operation we have

$$x(t) = Am(t)\cos(2\pi f_c t) + n(t)$$

where $n(t)$ represents bandpass filtered white Gaussian noise centered at $\pm f_c$ with spectral height $\frac{N_0}{2}$ and bandwidth B. After mixing and passing through the LPF we get an output consisting of signal and noise components. Define the SNR_{out} at the output as the power in the signal component divided by the power in the noise component. Similarly, define SNR_{in} at the input (before the mixing) as the power in the signal component divided by the power in the noise component for the signal at that point. Compute SNR_{out} in terms of SNR_{in}.

Solution. Write $n(t) = n_c(t)\cos(2\pi f_c t) - n_s(t)\sin(2\pi f_c t)$ we get

$$\begin{aligned} z(t) &= \frac{A^2}{2} m(t)\cos(\theta) + \frac{A^2}{2} m(t)\cos(4\pi f_c t + \theta) \\ &+ \frac{n_c(t)}{2}\cos(\theta) + \frac{n_c(t)}{2}\cos(4\pi f_c t + \theta) \\ &- \frac{n_s(t)}{2}\sin(\theta) - \frac{n_s(t)}{2}\sin(4\pi f_c t + \theta). \end{aligned}$$

We assume the bandpass filtered noise occupies the same bandwidth as $m(t)$ so at the output of the LPF we get

$$y(t) = \frac{A^2}{2} m(t)\cos(\theta) + \frac{n_c(t)}{2}\cos(\theta) - \frac{n_s(t)}{2}\sin(\theta).$$

We now make use of the fact that $E[n_c(t)n_s(t)] = E[n_c(t)]E[n_s(t)] = 0$ and $P_{n_c} = P_{n_s} = P_n$ we find

$$P_{out,signal} = \frac{A^2}{4} P_m \cos^2(\theta), \quad P_{out,noise} = \frac{P_n}{4}$$

and thus

$$SNR_{out} = \frac{A^2 P_m}{P_n} \cos^2(\theta).$$

Now

$$P_{in,signal} = \frac{A^2}{2} P_m, \quad P_{in,noise} = P_n$$

so

$$SNR_{in} = \frac{A^2 P_m}{2P_n}.$$

Therefore,

$$\frac{SNR_{out}}{SNR_{in}} = 2\cos^2(\theta).$$

Note: If the receiver has knowledge of θ so that, in effect, $\theta = 0$ in the above analysis, then $\frac{SNR_{out}}{SNR_{in}} = 2$. This is a well known result in communication systems. (See, for example, Communication Systems by Simon Haykin, Wiley,1983, page 325. Haykin does not provide the analysis we did here but does state our result. He does show, however, that the ratio of the output SNR to the channel SNR is unity. This is equivalent to what we did in class when comparing the demodulator output to a baseband system.)

Problem 3. A carrier wave of the form $c(t) = 40\cos(2\pi f_c t)$ of frequency $f_c = 50$ MHz is frequency-modulated by a message waveform of amplitude 10 volts and whose highest frequency content is 200 Hz. The modulation index is 10.

a. Write an expression for the modulated signal $x(t)$.

   Solution:
$$x(t) = A_c \cos\left[2\pi f_c t + 2\pi k_f \int_0^t m(s)ds\right]$$
   where $A_c = 40$, $f_c = 50$ MHz and $k_f = 200$ (see below). Note you cannot assume $m(t)$ is sinusoidal or any particular form and compute its integral in this expression.

b. What is the maximum frequency deviation of the modulated signal?

   Solution: $\Delta f = 200(10) = 2000$ Hz. Also, $\Delta f = k_f A_m$ so $k_f = 2000/10 = 200$ Hz/volt.

c. What is the power content of the modulated signal?

   Solution:
$$P_x = \frac{A_c^2}{2} = 800 \text{ Watts.}$$

d. Find the bandwidth of the modulated signal using Carson's Rule.

   Solution: $B \approx 2\Delta f + 2W = 4000 + 400 = 4400$ Hz.

Problem 4. The wide-sense stationary process $X(t)$ has a correlation function denoted by $R_X(\tau)$ and a power spectral density denoted by $S_X(f)$. Suppose we form a new random process as

$$Y(t) = X(t-T) - X(t-2T)$$

where T is some fixed positive constant.

a. Find the correlation function for $Y(t)$. Is $Y(t)$ wide-sense stationary?

Solution:

$$\begin{aligned} R_Y(t_1, t_2) &= E\left[Y(t_1)Y(t_2)\right] \\ &= E\left[(X(t_1-T) - X(t_1-2T))\,(X(t_2-T) - X(t_2-2T))^*\right] \\ &= R_X(t_1-t_2) - R_X(t_1-t_2+T) - R_X(t_1-t_2-T) + R_X(t_1-t_2). \end{aligned}$$

Let $\tau = t_1 - t_2$. Then,

$$R_Y(\tau) = 2R_X(\tau) - R_X(\tau+T) - R_X(\tau-T).$$

Clearly, $Y(t)$ is WSS.

b. Find the power spectral density of $Y(t)$. (Note: depending on how you work part (b) it can be worked independent of part (a).) Simplify your answer as much as possible.

Solution: Taking the Fourier transform of $R_Y(\tau)$ we get

$$S_Y(f) = 2S_X(f) - e^{j2\pi fT}S_X(f) - e^{-j2\pi fT}S_X(f)$$

which simplifies to

$$S_Y(f) = 2S_X(f)\left[1 - \cos(2\pi fT)\right].$$

Problem 5. Consider the reception of a BPSK type signal with no noise of the form

$$r(t) = A\cos(2\pi f_c t + \phi)$$

where A has the same sign for T seconds and ϕ is some unknown phase value. In class we showed how to process this type of signal (for $\phi = 0$) to determine the information (i.e., the sign of A) by mixing this signal with a locally generated sinusoid and then integrating for T seconds (lowpass filtering) before making our decision. Suppose at the receiver we know the exact value of f_c but we think $\phi = 0$. Hence, ϕ represents a phase error. If $T = 1$ second find the largest value of $|\phi|$ so that the power in the signal after low-pass filtering is not degraded more than 1 dB relative to the case when $\phi = 0$.

Note: You may assume that $f_c \gg 1/T$ so that after lowpass filtering any terms containing sinusoids with high frequencies may be ignored.

Solution: We see that the signal we get after mixing and integrating for 1 second is

$$\tilde{r} = \frac{A}{2}\cos(\phi).$$

To find the power degradation we compute the power in the output relative to the power of the input to get

$$\frac{(A^2/4)\cos^2(\phi)}{A^2/4} = \cos^2(\phi).$$

We set this last result equal to 1 dB degradation (or -1 dB) and get

$$\cos^2(\phi) = 10^{-1/10} \Rightarrow \phi = 0.4707 \ \text{ radians} = 26.97^o.$$

Problem 6. Let $\hat{s}(t)$ denote the Hilbert transform of $s(t)$.

a. Show how you can combine $s(t)$ and $\hat{s}(t)$ in such a way to form $z(t)$, where $z(t)$ is complex and has the property that its Fourier transform suppresses negatives frequencies and amplifies the non-negative frequencies.

   Solution: We first note that

$$\hat{S}(f) = \begin{cases} -jS(f), & f \geq 0 \\ jS(f), & f < 0. \end{cases}$$

   Then if we let $Z(f) = S(f) + j\hat{S}(f)$ we get

$$Z(f) = \begin{cases} 2S(f), & f \geq 0 \\ 0, & f < 0. \end{cases}$$

   Hence,

$$z(t) = s(t) + j\hat{s}(t)$$

   works for us.

b. Find an explicit expression for your $z(t)$ when $s(t) = \cos(2\pi f_c t)$.

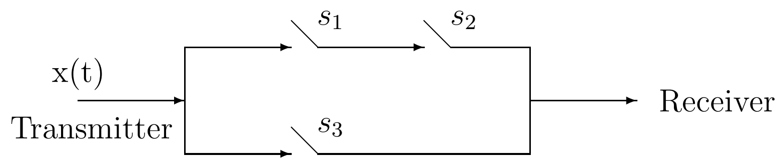
   Solution: Since $s(t) = \cos(2\pi f_c t)$ we get $\hat{s}(t) = \sin(2\pi f_c t)$. Therefore,

$$z(t) = \cos(2\pi f_c t) + j \sin(2\pi f_c t$$

   or

$$z(t) = e^{j2\pi f_c t}.$$

Problem 7. Consider the transmission of a signal as shown in the following diagram.



A signal is transmitted along two paths as shown. In the upper path there are two switches to pass through while in the lower path there is one switch to pass through. Each switch s_i operates independently and allows the signal to pass with probability p_i for $i = 1, 2, 3$. The signal transmission is successful if the signal $x(t)$ sent at the transmitter reaches the receiver along either or both paths. Find the probability that the transmission is successful if

a. $p_1 = 2/3,\ p_2 = 2/3,\ p_3 = 1/2$.

Solution: Let U be the event of signal passing thru upper path and L be the event of signal passing thru lower path. Then,

$$\begin{aligned} P(success) &= P(U) + P(L) - P(U)P(L) \\ &= p_1 p_2 + p_3 - p_1 p_2 p_3 = \frac{13}{18}. \end{aligned}$$

b. $p_1 = 2/3,\ p_2 = 2/3,\ p_3$ is a discrete random variable with probability mass function

$$P\left(p_3 = \frac{3}{4}\right) = \frac{1}{2}, \quad P\left(p_3 = 0\right) = \frac{1}{2}.$$

Solution:

$$\begin{aligned} &P(success) \\ &= P(U) + P(L|p_3 = 3/4)P(p_3 = 3/4) + P(L|p_3 = 0)P(p_3 = 0) \\ &-P(U)\left(P(L|p_3 = 3/4)P(p_3 = 3/4) + P(L|p_3 = 0)P(p_3 = 0)\right) \\ &= \frac{47}{72}. \end{aligned}$$