EE 567

Handout 4

Sampling Theorem for Bandlimited WSS Random Processes

Definition: A WSS random process $X(u, T)$ is said to be *bandlimited* to $[\omega_1, \omega_2]$ if $S_X(\omega) = 0$ for $\omega \notin [\omega_1, \omega_2]$.

If $\omega_1 = 0$, we have a lowpass system with cutoff frequency $\omega_c = \omega_2$. Here

$$R_X(\tau) = \sum_{n=-\infty}^{\infty} R_X(nT) \frac{\sin[\omega_c(\tau - nT)]}{\omega_c(\tau - nT)}$$

where $T = \pi/\omega_c$.

Theorem: If a 2nd order WSS RP $X(u, t)$ is lowpass with cutoff frequency ω_c, then

$$X(u, t) = \sum_{n=-\infty}^{\infty} X(u, nT) \frac{\sin[\omega_c(t - nT)]}{\omega_c(t - nT)} \quad (M.S.)$$

i.e., with

$$X_N(u, t) = \sum_{n=-N}^{N} X(u, nT) \frac{\sin[\omega_c(t - nT)]}{\omega_c(t - nT)}$$

then

$$\lim_{N\to\infty} E[|X(u, t) - X_N(u, t)|^2] = 0.$$

Proof:

$$E[|X(u, t) - X_N(u, t)|^2]$$
$$= E[(X(u, t) - X_N(u, t))X^*(u, t)] - E[(X(u, T) - X_N(u, t))X_N^*(u, t)].$$

Now

$$E[(X(u, t) - X_N(u, t))X^*(u, t)]$$
$$= R_X(0) - \sum_{n=-N}^{N} R_X(nT - t) \frac{\sin[\omega_c(t - nT)]}{\omega_c(t - nT)}.$$

But

$$R_X(\tau - t) = \sum_{n=-\infty}^{\infty} R_X(nT - t) \frac{\sin[\omega_c(\tau - nT)]}{\omega_c(\tau - nT)}.$$

Set $\tau = t$ to get

$$R_X(0) = \sum_{n=-\infty}^{\infty} R_X(nT - t)\frac{\sin[\omega_c(t - nT)]}{\omega_c(t - nT)}.$$

Thus

$$E[(X(u,t) - X_N(u,t))X^*(u,t)] \to 0 \text{ as } N \to \infty.$$

Now

$$E[(X(u,t) - X_N(u,t))X^*(u,mT)]$$
$$= R_X(t - mT) - \sum_{n=-N}^{N} R_X(nT - mT)\frac{\sin[\omega_c(t - nT)]}{\omega_c(t - nT)}.$$

But

$$R_X(\tau - mT) = \sum_{n=-\infty}^{\infty} R_X(nT - mT)\frac{\sin[\omega_c(\tau - nT)]}{\omega_c(\tau - nT)}.$$

Set $\tau = t$ to get

$$R_X(t - mT) = \sum_{n=-\infty}^{\infty} R_X(nT - mT)\frac{\sin[\omega_c(t - nT)]}{\omega_c(t - nT)}.$$

Thus

$$E[(X(u,t) - X_N(u,t))X^*(u,mT)] \to 0 \text{ as } N \to \infty.$$

Next let

$$\hat{X}(u,t) = \lim_{N\to\infty} X_N(u,t) \quad (M.S.).$$

Then

$$E[(X(u,t) - \hat{X}(u,t))X^*(u,mT)] = 0.$$

But, $X_N(u,t)$ is a linear combination of $X(u,mT)$ for $m = -N$ to N so

$$E[(X(u,t) - \hat{X}(u,t))X_N^*(u,t)] = 0$$

which implies

$$\lim_{N\to\infty} E[(X(u,t) - X_N(u,t))X_N^*(u,t)] = 0.$$

So

$$\lim_{N\to\infty} E[|X(u,t) - X_N(u,t)|^2] = 0.$$