EE 567
Homework 5 solution

Problem 1. Consider a channel in which the output signal, $v_o(t)$, is related to the input signal, $v_i(t)$, via

$$v_o(t) = a_1 v_i(t) + a_2 v_i^2(t) + a_3 v_i^3(t)$$

where a_1, a_2 and a_3 are constants. Let

$$v_i(t) = A_c cos[2\pi f_c t + \phi(t)]$$

where

$$\phi(t) = 2\pi k_f \int_0^t m(\tau)\, d\tau$$

a. Show that the channel output $v_o(t)$ contains a dc component and three frequency modulated waves with carrier frequencies f_c, $2f_c$ and $3f_c$. (10)

b. To extract an FM wave that is the same as that at the channel input, except possibly for a change in carrier amplitude, show that by using Carson's rule the carrier frequency, fc, must satisfy the following condition:

$$f_c > 3\Delta f + 2W$$

where W is the highest frequency component of the message m(t) and Δf is the frequency deviation of the FM wave $v_i(t)$. (10)

Solution:

a. Given that $v_i(t) = A_c cos[2\pi f_c t + \phi(t)]$, we have the output signal

$$\begin{aligned} v_o(t) &= a_1 v_i(t) + a_2 v_i^2(t) + a_3 v_i^3(t) \\ &= a_1 A_c cos[2\pi f_c t + \phi(t)] + a_2 A_c^2 \cos^2[2\pi f_c t + \phi(t)] + \\ &\quad a_3 A_c^3 \cos^3[2\pi f_c t + \phi(t)] \\ &= a_1 A_c cos[2\pi f_c t + \phi(t)] + \frac{a_2 A_c^2}{2}(cos[4\pi f_c t + 2\phi(t)] + 1) + \\ &\quad \frac{a_3 A_c^3}{4}(cos[6\pi f_c t + 3\phi(t)] + 3\, cos[2\pi f_c t + \phi(t)]) \\ &= \frac{a_2 A_c^2}{2} + \left(a_1 A_c + \frac{3a_3 A_c^3}{4}\right) cos[2\pi f_c t + \phi(t)] + \frac{a_2 A_c^2}{2} cos[4\pi f_c t + \\ &\quad 2\phi(t)] + \frac{a_3 A_c^3}{4}\cos[6\pi f_c t + 3\phi(t)] \end{aligned}$$

which contains a DC component and three FM waveforms with carrier frequencies at f_c, $2f_c$ and $3f_c$.

b. Carson's rule tells us that effective bandwidth of a single-tone FM waveform is roughly $2(\Delta f + f_m)$ where Δf is the frequency deviation and f_m is the frequency of the message signal. Additionally, if the signal $\phi(t)$ is not a single-tone, then the effective bandwidth is estimated to be $2(\Delta f + W)$ by a worst-case single-tone FM with $f_m = W$ where W is frequency of the highest frequency component in $\phi(t)$. Therefore, the effective bandwidths of the three FM components in $v_o(t)$ are

$$B_1 = 2\Delta f + 2W$$
$$B_2 = 4\Delta f + 2W$$
$$B_3 = 6\Delta f + 2W$$

Notice that the coefficients in front of $\phi(t)$ do not change the highest frequency component of $\phi(t)$, but change the frequency deviations of the FM waveforms.

To extract an FM waveform at the output that is the same as the channel input $v_i(t) = A_c cos[2\pi f_c t + \phi(t)]$, we need to make sure that the first FM component of $v_o(t)$ is not corrupted by the second FM component, third FM component and DC component. Thus, we want

$$f_c > \Delta f + W\text{; not corrupted by DC}$$
$$2f_c - f_c > (2\Delta f + W) + (\Delta f + W)\text{; not corrupted by second FM}$$
$$3f_c - f_c > (3\Delta f + W) + (\Delta f + W)\text{; not corrupted by third FM}$$

we conclude that we require $f_c > 3\Delta f + 2W$.

Problem 2. A certain bandpass system has an FM wave given by

$$s(t) = A_c sin[2\pi f_c t + \beta sin(2\pi f_m t)]$$

where f_c is much greater than f_m. Find the equivalent low-pass signal representation for this bandpass signal. (20)

Solution:

$$\begin{aligned} s(t) &= A_c sin[2\pi f_c t + \beta sin(2\pi f_m t)] \\ &= A_c sin(2\pi f_c t) cos(\beta sin(2\pi f_m t)) + A_c cos(2\pi f_c t) sin(\beta sin(2\pi f_m t)) \\ &= \Re\{m(t) e^{j2\pi f_c t}\} \end{aligned}$$

where the low-pass complex envelope $m(t) = \mathrm{A_c}[sin(\beta sin(2\pi f_m t)) - jcos(\beta sin(2\pi f_m t))]$

Problem 3. Let $\hat{s}(t)$ denote the Hilbert transform of $s(t)$.

a. Show $\hat{\hat{s}}(t) = - s(t)$. (10)

b. If $s(t) = cos(w_0 t)$ then show $\hat{s}(t) = sin(w_0 t)$. (5)

c. Show that a real signal $s(t)$ and its Hilbert transform are orthogonal, i.e., show: (5)

$$\int_{-\infty}^{\infty} \hat{s}(t) s(t) dt = 0$$

Solution:

a. The Fourier transform of a Hilbert transform system is $H(f) = -jsgn(f)$. We find that $\hat{\hat{S}}(f) = S(f) H^2(f)$, hence, $\hat{\hat{s}}(t) = - s(t)$.

b. Using the same trick as part a., we have

$$\hat{S}(f) = S(f)H(f)$$
$$= \frac{1}{2}(\delta(f - \frac{w_0}{2\pi}) + \delta(f + \frac{w_0}{2\pi}))(-jsgn(f))$$
$$= \frac{1}{2j}(\delta(f - \frac{w_0}{2\pi}) - \delta(f + \frac{w_0}{2\pi}))$$

Therefore, $\hat{s}(t) = sin(w_0 t)$.

c. Notice that the Hilbert transform of a signal $s(t)$ can be also looked as convolution between $s(t)$ and $h(t) = 1/\pi t$ in time domain, i.e.,

$$\hat{s}(t) = s(t) * h(t)$$
$$= \int_{-\infty}^{\infty} s(\tau)h(t-\tau)d\tau$$
$$= \int_{-\infty}^{\infty} \frac{s(\tau)}{\pi(t-\tau)} d\tau$$

The integral can be written as

$$\int_{-\infty}^{\infty} \hat{s}(t)s(t)dt = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{s(\tau)}{\pi(t-\tau)} s(t)d\tau dt$$
$$= \int_{-\infty}^{\infty} s(\tau) \int_{-\infty}^{\infty} \frac{s(t)}{\pi(t-\tau)} dt\, d\tau$$
$$= -\int_{-\infty}^{\infty} s(\tau)\hat{s}(\tau)d\tau$$

Hence,

$$\int_{-\infty}^{\infty} \hat{s}(t)s(t)dt = 0$$

Problem 4. In order for our linear PLL model to be valid we required $sin\phi_e(t) \approx \phi_e(t)$. As a rule of thumb we take this approximation as acceptable if $|\phi_e(t)| \leq \pi/6\, rad\ (= 30\ degrees)$. At any instant of time, $\phi_e(t)$ is a Gaussian random variable with mean 0 and variance σ_e^2. Therefore, the probability that the mean process is less than any particular phase value θ_0 at any particular time is

$$P(|\phi_e(t)| < \theta_0) = \int_{-\theta_0}^{\theta_0} \frac{1}{\sqrt{2\pi}\sigma_e} exp(-\frac{x^2}{2\sigma_e^2})dx$$

One can consider this probability (times 100) as the percent of time the loop remains in linear operation.

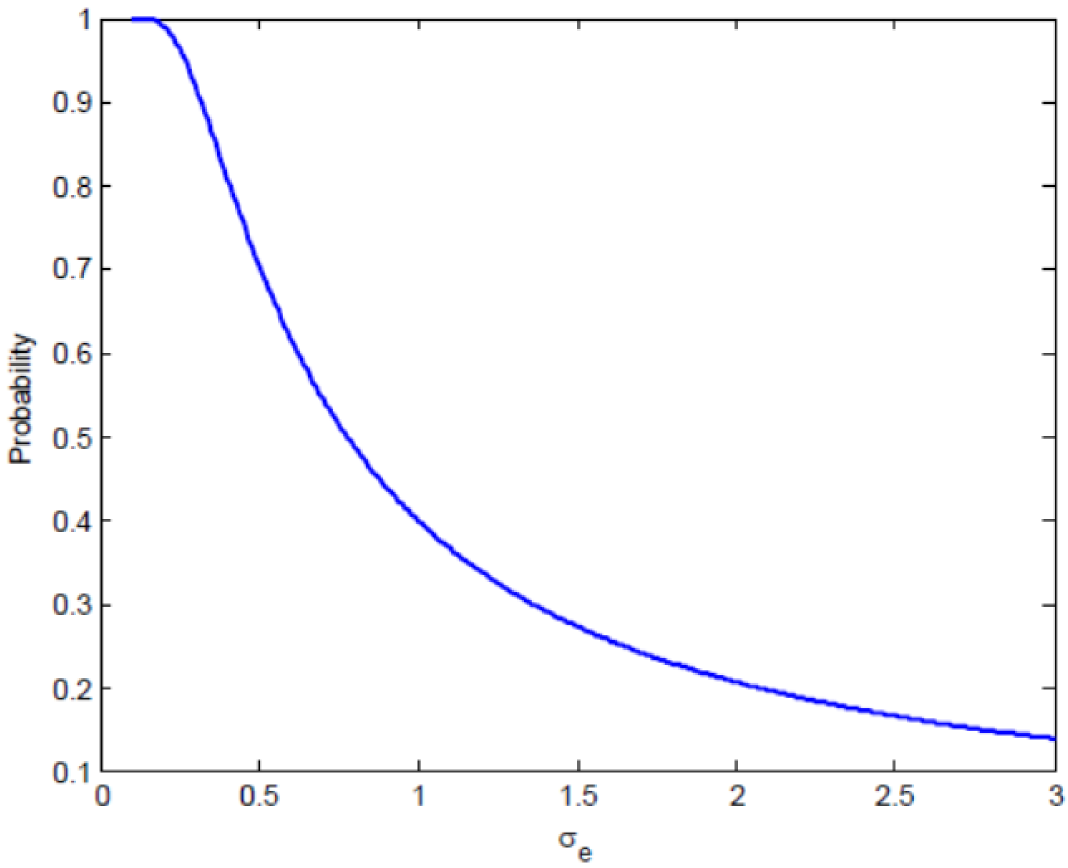
a. Using $\theta_0 = \pi/6$ plot $P(|\phi_e(t)| < \theta_0)$ as a function of σ_e.

b. Find an upper bound on σ_e so that the loop is linear at least 95% of the time.

Solution

a. Please see the figure below. (10)

b. σ_e needs to approximately less than 0.267 so that the loop is linear with probability at least 0.95 (Other numbers that are close to this result could also be considered as correct). (10)



Problem 5. Consider a first-order PLL as described in class, i.e., the loop filter is $H(f) = 1$. Then

$$\Phi_e(f) = \frac{1}{1 + K_0/jf}\Phi_1(f)$$

We wish to investigate the loop behavior in the presence of a frequency modulated input. We have

$$m(t) = A_m cos(2\pi f_m t)$$

with the corresponding FM wave given by

$$s(t) = A_c sin[2\pi f_c t + \beta sin(2\pi f_m t)]$$

where β is the modulation index.

a. Give the equation for $\Phi_1(t)$. (5)

b. Let us write

$$\Phi_e(t) = \Phi_{e0} cos(2\pi f_m t + \psi)$$

Give the equations for Φ_{e0} and ψ. In your expression for Φ_{e0} use the variable $\Delta f = \beta f_m$. (5)

c. Plot the phase error amplitude Φ_{e0} normalized with respect to $\Delta f / K_0$ versus the dimensionless parameter f_m / K_0. (5)

d. For the loop to track the frequency modulation closely we require the phase error $\Phi_e(t)$ to remain within the linear region which implies $\sin[\Phi_e(t)]$

$\approx \Phi_e(t)$ which is true if $\Phi_e(t) \leq 0.5$ radians. Explain why this requires $\Delta f \leq 0.5\mathrm{K}_0$. (5)

Solution:

a. Given the modulating signal $m(t) = A_m cos(2\pi f_m t)$, the frequency sensitivity K_f , and the FM signal $s(t) = A_c sin[2\pi f_c t + \phi_1(t)]$, we have

$$\phi_1(t) = 2\pi K_f \int_0^t m(s)ds$$

$$= \frac{K_f A_m}{f_m} sin(2\pi f_m t)$$

$$= \beta sin(2\pi f_m t)$$

b. Let us write$\Phi_e(t) = \Phi_{e0} cos(2\pi f_m t + \psi)$, then

$$\Phi_e(f) = \frac{\phi_{e0}\cos(\psi)}{2}[\delta(f - f_m) + \delta(f + f_m)] - \frac{\phi_{e0}\sin(\psi)}{2j}[\delta(f - f_m) - \delta(f + f_m)]$$

In addition, from the first order model of PLL

$$\Phi_e(f) = \frac{1}{1+K_0/jf}\Phi_1(f)$$

With $\Phi_1(f) = \frac{\beta}{2j}[\delta(f - f_m) - \delta(f + f_m)]$, we have

$$\Phi_e(f) = \frac{1}{1+\frac{K_0}{jf}}\frac{\beta}{2j}[\delta(f - f_m) - \delta(f + f_m)]$$

$$= \frac{\beta K_0/2f - j\beta/2}{1+(K_0/f)^2}[\delta(f - f_m) - \delta(f + f_m)]$$

We then solve the follow system of equations

$$\begin{cases} \phi_{e0} cos(\psi) = \frac{\beta K_0/f_m}{1+(K_0/f_m)^2} \\ \phi_{e0} sin(\psi) = \frac{-\beta}{1+(K_0/f_m)^2} \end{cases}$$

Thus, we have

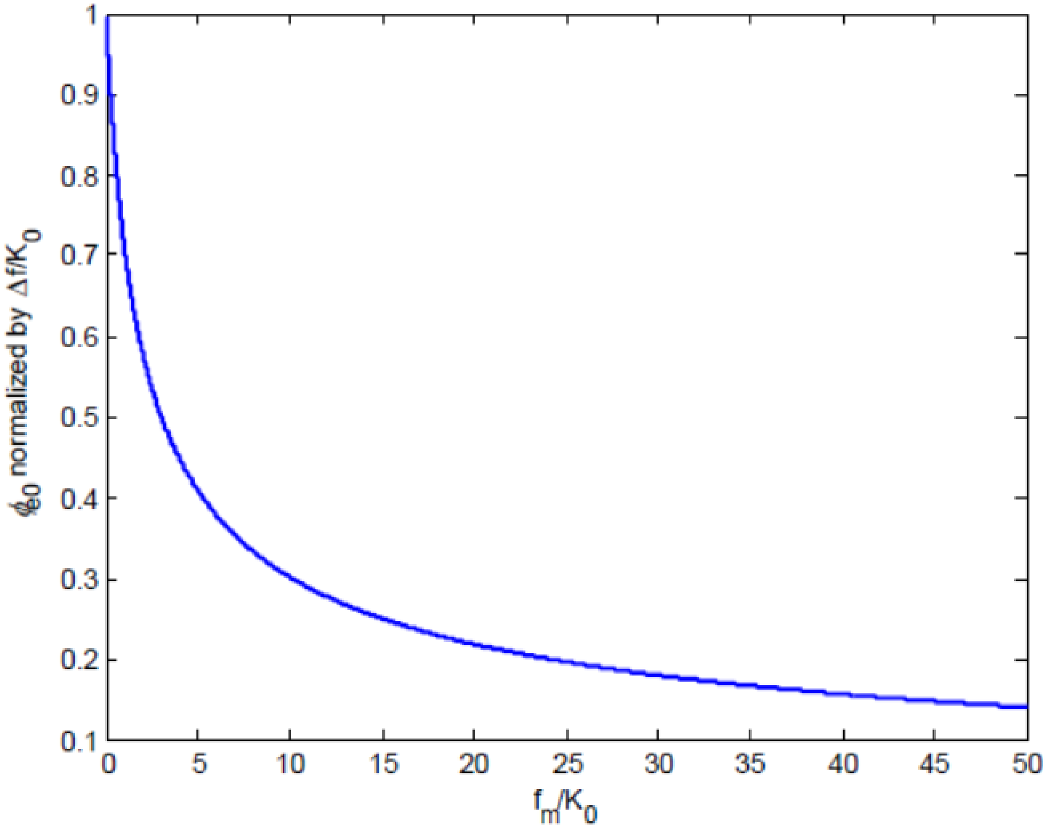
$$tan(\psi) = -\frac{f_m}{K_0} \Rightarrow \psi = tan^{-1} - \frac{f_m}{K_0}$$

$$\cos(\psi) = \frac{K_0}{\sqrt{K_0^2 + f_m^2}}$$

$$\phi_{e0} = \frac{\Delta f}{\sqrt{K_0^2 + f_m^2}}$$

c. Please see the figure below.

d. Since we require $|\Phi_e(t)| \leq 0.5$ for the approximation implies $\sin[\Phi_e(t)] \approx \Phi_e(t)$ and $\Phi_e(t) = \Phi_{e0} cos(2\pi f_m t + \psi)$, this implies $\Delta f/\mathrm{K}_0 \leq 0.5$ is sufficient for the approximation.



1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
φe0 normalized by Δf/K0
0
5
10
15
20
25
30
35
40
45
50
fm/K0