EE 567

Homework 5

Due Tuesday, October 2, 2018

Work all 5 problems.

Problem 1. Consider a channel in which the output signal, $v_o(t)$, is related to the input signal, $v_i(t)$, via

$$v_o(t) = a_1 v_i(t) + a_2 v_i^2(t) + a_3 v_i^3(t)$$

where $a_1,\ a_2$ and a_3 are constants. Let

$$v_i(t) = A_c \cos[2\pi f_c t + \phi(t)]$$

where

$$\phi(t) = 2\pi k_f \int_0^t m(\tau) d\tau.$$

a. Show that the channel output $v_o(t)$ contains a dc component and three frequency modulated waves with carrier frequencies $f_c,\ 2f_c$ and $3f_c$.

b. To extract an FM wave that is the same as that at the channel input, except possibly for a change in carrier amplitude, show that by using Carson's rule the carrier frequency, f_c, must satisfy the following condition:

$$f_c > 3\Delta f + 2W$$

where W is the highest frequency component of the message $m(t)$ and Δf is the frequency deviation of the FM wave $v_i(t)$.

Problem 2. A certain bandpass system has an FM wave given by

$$s(t) = A_c \sin[2\pi f_c t + \beta \sin(2\pi f_m t)]$$

where f_c is much greater than f_m. Find the equivalent low-pass signal representation for this bandpass signal.

Problem 3. Let $\hat{s}(t)$ denote the Hilbert transform of $s(t)$.

a. Show $\hat{\hat{s}}(t) = -s(t)$.

b. If $s(t) = \cos(\omega_0 t)$ then show $\hat{s}(t) = \sin(\omega_0 t)$.

c. Show that a real signal $s(t)$ and its Hilbert transform are orthogonal, i.e., show

$$\int_{-\infty}^{\infty} s(t)\hat{s}(t)dt = 0.$$

Problem 4. In order for our linear PLL model to be valid we required $\sin \phi_e(t) \approx \phi_e(t)$. As a rule of thumb we take this approximation as acceptable if $|\phi_e(t)| \leq \pi/6$ rad (=30 degrees). At any instant of time, $\phi_e(t)$ is a Gaussian random variable with mean 0 and variance σ_e^2. Therefore, the probability that the mean process is less than any particular phase value θ_0 at any particular time is

$$P\left(|\phi_e(t)| < \theta_0\right) = \int_{-\theta_0}^{\theta_0} \frac{1}{\sqrt{2\pi}\sigma_e} \exp\left(-\frac{x^2}{2\sigma_e^2}\right) dx.$$

One can consider this probability (times 100) as the percent of time the loop remains in linear operation.

a. Using $\theta_0 = \pi/6$ plot $P\left(|\phi_e(t)| < \theta_0\right)$ as a function of σ_e.

b. Find an upper bound on σ_e so that the loop is linear at least 95% of the time.

Problem 5. Consider a first-order PLL as described in class, i.e., the loop filter is $H(f) = 1$. Then

$$\Phi_e(f) = \frac{1}{1 + K_0/jf}\Phi_1(f).$$

We wish to investigate the loop behavior in the presence of a frequency modulated input. We have

$$m(t) = A_m \cos(2\pi f_m t)$$

with the corresponding FM wave given by

$$s(t) = A_c \sin[2\pi f_c t + \beta \sin(2\pi f_m t)]$$

where β is the modulation index.

a. Give the equation for $\phi_1(t)$.

b. Let us write

$$\phi_e(t) = \phi_{e0} \cos(2\pi f_m t + \psi).$$

Give the equations for ϕ_{e0} and ψ. In your expression for ϕ_{e0} use the variable $\Delta f = \beta f_m$.

c. Plot the phase error amplitude ϕ_{e0} normalized with respect to $\Delta f / K_0$ versus the dimensionless parameter f_m / K_0.

d. For the loop to track the frequency modulation closely we require the phase error $\phi_e(t)$ to remain within the linear region which implies $\sin[\phi_e(t)] \approx \phi_e(t)$ which is true if $\phi_e(t) \leq 0.5$ radians. Explain why this requires $\Delta f \leq 0.5 K_0$.