EE 567

Homework 1 solution

Problem 1. Compute the Fourier transform of $\mathrm{R}_2(t)$ defined by

$$\mathrm{R}_2(t) = \begin{cases} 1 & -2 \le t \le -1 \\ 1 & 1 \le t \le 2 \\ 0 & elsewhere \end{cases}$$

and simplify your answer to terms involving sinusoidal functions (i.e., with- out complex exponentials). Show all your work. Sketch a plot of its graph in the frequency domain. You should produce two plots (one for the magnitude of the Fourier transform and one for the phase). You may use Matlab to produce the plots. (20 points)

Hint: It is quicker to utilize Fourier transform properties.

Solution:

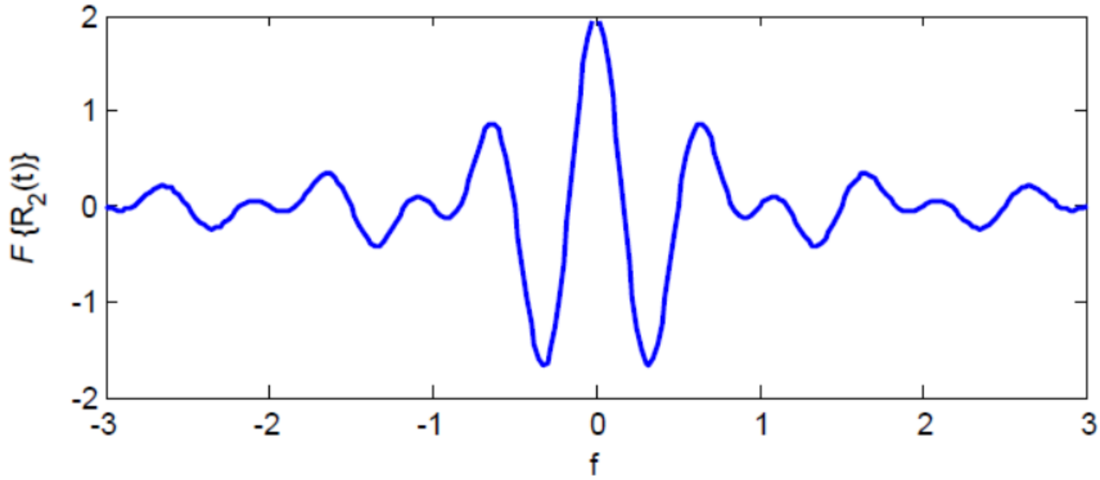
Notice that a unit square pulse $\mathrm{p}(t)$ has Fourier transform $\mathrm{P}(f) = \frac{\sin\pi f}{\pi f}$,where

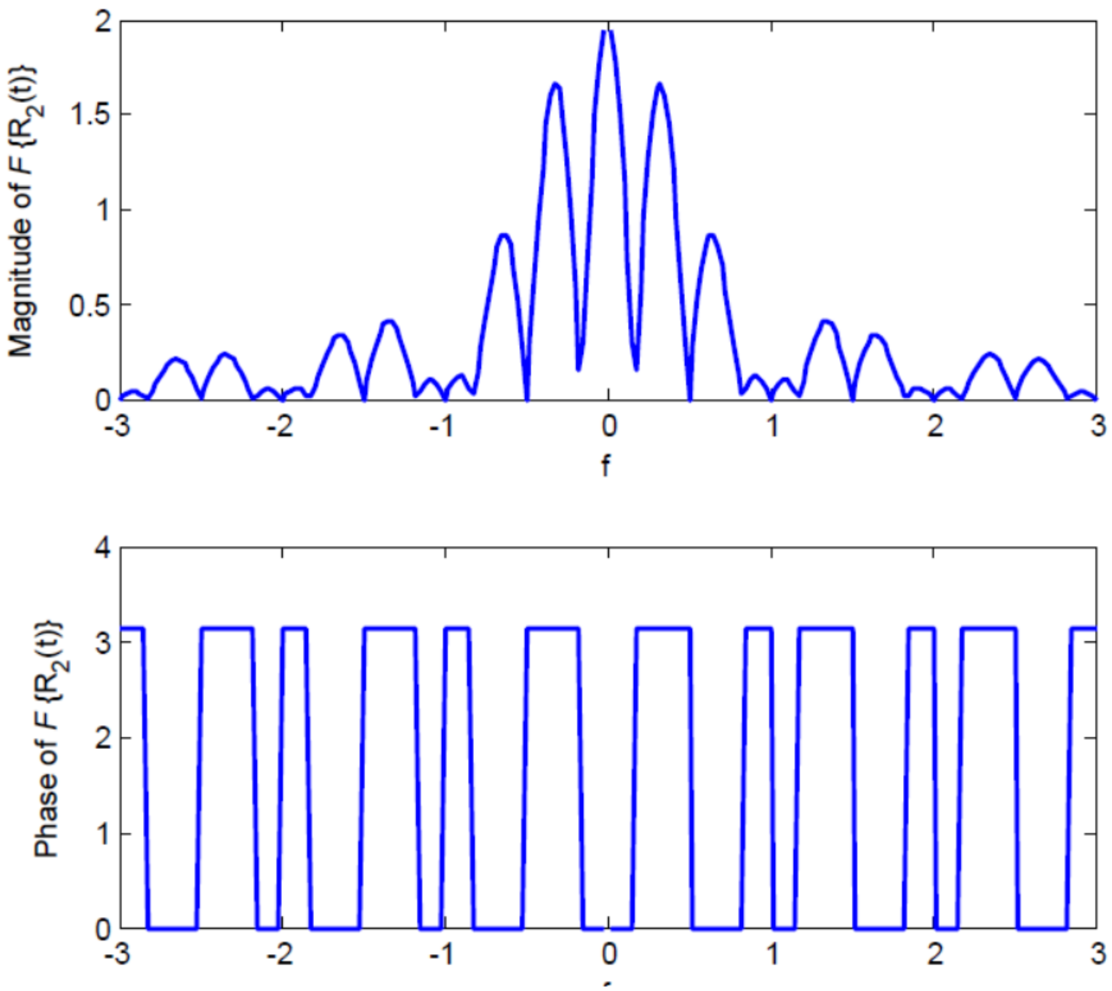
$$\mathrm{p}(t) = \begin{cases} 1 & if -\frac{1}{2} \le t \le \frac{1}{2} \\ 0 & otherwise \end{cases}$$

With $\mathrm{R}_2(t) = p\left(t - \frac{3}{2}\right) + p(t + \frac{3}{2})$, we can compute the Fourier transform of $\mathrm{R}_2(t)$ using time-shifting property $\mathrm{p}(t - \tau) \leftrightarrow \mathrm{P}(f)\mathrm{e}^{-j2\pi f\tau}$

$$F\{\mathrm{R}_2(t)\} = \mathrm{P}(f)\mathrm{e}^{-j3\pi f} + \mathrm{P}(f)\mathrm{e}^{j3\pi f}$$

$$= \frac{2\cos(3\pi f)\sin\pi f}{\pi f}$$





Problem 2. Determine which of the following systems is linear.
a. $y(t) = x(t) + 1$. (5 points)
b. $y(t) = Asin\, x(t)$, A is a constant. (5 points)
c. $y(t) = x(at)$, a is a constant. (5 points)
d. $y(t) = x(t-1)$. (5 points)

Solution:
A system is linear if it possesses additivity property and homogeneity property, that is, suppose we have input/output pairs $x_1(t) \rightarrow y_1(t)$ and $x_2(t) \rightarrow y_2(t)$, the output of the weighted sum of input $x(t) = \alpha x_1(t) + \beta x_2(t)$ is

$$y(t) = \alpha y_1(t) + \beta y_2(t)$$

which is the weighted sum of individual output.
a. Let $x(t) = \alpha x_1(t) + \beta x_2(t)$, then the corresponding output

$$\begin{aligned} y(t) &= x(t) + 1 \\ &= \alpha x_1(t) + \beta x_2(t) + 1 \\ &\neq \alpha y_1(t) + \beta y_2(t) \end{aligned}$$

Hence, it is not a linear system.

b. Let $x(t) = \alpha x_1(t) + \beta x_2(t)$, then

$$\begin{aligned} y(t) &= A\sin x(t) \\ &= \mathrm{A\sin}\left(\alpha x_1(t) + \beta x_2(t)\right) \\ &\neq \alpha y_1(t) + \beta y_2(t) \end{aligned}$$

Hence, it is not a linear system.

c. Let $x(t) = \alpha x_1(t) + \beta x_2(t)$, then

$$\begin{aligned} y(t) &= x(at) \\ &= \alpha x_1(at) + \beta x_2(at) \\ &= \alpha y_1(t) + \beta y_2(t) \end{aligned}$$

It is a linear system.

d. Let $x(t) = \alpha x_1(t) + \beta x_2(t)$, then

$$\begin{aligned} y(t) &= x(t-1) \\ &= \alpha x_1(t-1) + \beta x_2(t-1) \\ &= \alpha y_1(t) + \beta y_2(t) \end{aligned}$$

It is a linear system.

Problem 3. In class we sometimes move the limit operation inside or outside the integral. This problem illustrates this is not always valid (there are theorems in integration theory that specify when this is valid). Let

$$f_n(t) = nt(1-t^2)^n, 0 \le t \le 1.$$

a. Show $\lim\limits_{n\to\infty} f_n(t) = 0$ (8 points)
b. Show $\int_0^1 f_n(t)dt = \frac{n}{2(n+1)}$. (8 points)
c. Deduce $\lim\limits_{n\to\infty} \int_0^1 f_n(t)dt \neq \int_0^1 \lim\limits_{n\to\infty} f_n(t)dt$ (4 points)

Solution:

a. Observe that $f_n(t) = nt(1-t^2)^n = 0\ if\ t = 1$ or 0. For $t \in (0,1)$,

$$\begin{aligned} \lim_{n\to\infty} f_n(t) &= \lim_{n\to\infty} \frac{nt}{(1-t^2)^{-n}} \\ &= \lim_{n\to\infty} \frac{t}{-ln(1-t^2)(1-t^2)^{-n}} \quad \text{(L'Hospital)} \\ &= \frac{t}{-\ln(1-t^2)} \lim_{n\to\infty} \frac{1}{(1-t^2)^{-n}} \\ &= 0 \end{aligned}$$

Hence $\lim\limits_{n\to\infty} f_n(t) = 0, \forall t \in [0,1]$.

b.

$$\begin{aligned} \int_0^1 f_n(t)dt &= -\frac{n}{2(1+n)}(1-t^2)^{n+1}\Big|_{t=0}^{1} \\ &= \frac{n}{2(n+1)} \end{aligned}$$

c. From part a. and part b., we have

$$\lim_{n\to\infty} \int_0^1 f_n(t)dt = \lim_{n\to\infty} \frac{n}{2(n+1)} = \frac{1}{2}$$

$$\int_0^1 \lim_{n\to\infty} f_n(t)dt = \int_0^1 0 = 0$$

Thus $\lim_{n\to\infty} \int_0^1 f_n(t)dt \neq \int_0^1 \lim_{n\to\infty} f_n(t)dt$ which means that swapping integral and limit is not always valid. (Extra: It is valid if the sequence of functions $\{f_n(t)\}$ converges uniformly.)

Problem 4. Compute the Fourier transform of x(t) where

$$\mathrm{x(t)} = \frac{1}{\sqrt{2\pi}} \exp(-t^2/2)$$

Solution:

$$F\big(x(t)\big) = \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} exp\left(-\frac{t^2}{2} - jwt\right) dt$$

$$= \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{t^2+jwt}{2}\right) dt$$

$$= \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{\left(t+\frac{jw}{2}\right)^2+\frac{w^2}{4}}{2}\right) dt$$

$$= \exp\left(-\frac{\mathrm{w}^2}{2}\right) \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{\left(t+\frac{jw}{2}\right)^2}{2}\right) dt$$

$$= \exp\left(-\frac{\mathrm{w}^2}{2}\right)$$

Problem 5. As stated in class one of the Dirichlet conditions for the existence of the Fourier series for a function $x(t)$ is that the number of maxima and minima of $x(t)$ in a period is finite. For the Fourier transform this condition is replaced by the number of maxima and minima is finite in any finite interval. Give an example of a function that is continuous in the interval (0,1) but has an infinite number of maxima and minima in that interval and thus violates this condition. Prove that your function does indeed have an infinite number of maxima and minima in (0,1). (20 points)

Solution: (an example)

Let $x(t) = cos\,(1/t)$. Notice that $cos(t)$ is a continuous function with maximum value 1 at $\mathrm{t} = 2\mathrm{k}\pi, \mathrm{k}\epsilon\mathrm{Z}$ and minimum value -1 at $\mathrm{t} = (2\mathrm{k}+1)\pi, \mathrm{k}\epsilon\mathrm{Z}$. Since $1/t$ has range $(1, \infty)$ if $\mathrm{t}\epsilon(0,1), \mathrm{x(t)} = \cos\,(1/\mathrm{t})$ has infinite number of maxima and minima in (0,1).